B.Sc. Semester-IV Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 42113 **Course Code : SH/MTH/403/C-10**

Course Title : Ring Theory & Linear Algebra-I

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer any **five** from the following questions: $2\times5=10$

a) Define nilpotent element in a ring. Show that $\overline{6}$ is nilpotent in $(\mathbb{Z}_6, +_6, \cdot_6)$.

b) Show that the ring $\mathbb{Z}/8\mathbb{Z}$ is not an integral domain where $\mathbb{Z}$ being the set of integers.

c) Find IJ in the ring of integers where $I = 2\mathbb{Z}$ and $J = 4\mathbb{Z}$.

d) Let R and S be two rings and $f : R \to S$ be a homomorphism of R into S. Then show that kerf is an ideal of R.

e) Let V be a vector space over the field F and $B = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ be a basis for V. Let $x \in V$ such that $x = \sum_{i=1}^{n} c_i\alpha_i$ for some $c_1, c_2, ..., c_n \in F$. Then prove that this representation of x with respect to the basis B is unique.

[Turn Over]

f) Give an example of a finite dimensional vector space and an example of an infinite dimensional vector space over a finite field.

g) Let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x, y) = (x + y, x - y, y)$. Find dimension of KerT.

h) Let T be a linear operator on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ defined by $T(x,y) = (-y, x)$ for all $(x, y) \in \mathbb{R}^2$. Find the matrix representation of T with respect to the standard ordered basis.

UNIT-II

2. Answer any **four** from the following questions: $5\times4=20$

a) Let $R = M_2(\mathbb{C})$ and $S = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}$. Show that S is a subring of R. Also find $C(S)$ where $C(S) = \{ A \in R \mid AB = BA \text{ for all } B \in S \}$. 2+3

b) Let I denote the set of all polynomials in $\mathbb{Z}[x]$ with constant terms zero. Show that I is a prime ideal but not maximal in $\mathbb{Z}[x]$.

c) Let I and J be two ideals of a ring R. Then show that $R/(I \cap J)$ is isomorphic to a subring of $\left(\frac{R}{I}\right) \times \left(\frac{R}{J}\right)$.

d) i) Consider the space $\mathbb{C}^2$ over the field $\mathbb{R}$. Find the coordinates of $(2+3i, 4-5i) \in \mathbb{C}^2$ with respect to its standard basis.

ii) Find the change of coordinate matrix for the ordered bases {(1,1,0),(1,0,1), (0,1,1)} and {(1,0,0), (1,1,0), (1,1,1)} of the real space $\mathbb{R}^3$. 2+3

e) If the linear transformation T on a vector space V(F) such that $T^2 - T + I = 0$, then show that T is invertible.

f) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by $T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$ for all (x, y, z) $\in \mathbb{R}^3$. Let B be the standard ordered basis of $\mathbb{R}^3$. If $B_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis of $\mathbb{R}^3$ where $\alpha_1 = (1,0,1), \alpha_2 = (-1,2,1), \alpha_3 = (2,1,1)$ then find an invertible matrix P such that $[T]_{B_1} = P^{-1}[T]_B P$ where $[T]_{B_1}$ stands for the matrix representation of T with respect to B_1.

UNIT-III

3. Answer any **one** of the following questions: 10×1=10

a) i) Give an example (with proper justification) of a ring R with 1 having a subring S such that S does not contain 1.

ii) Are the rings $(7\mathbb{Z},+,.)$ and $(16\mathbb{Z},+,.)$ isomorphic? Justify your answer.

iii) Find $k \in \mathbb{R}$ such that the set S = {(k, k, 1), (k, 1, k), (1, k, k)} becomes linearly independent in the vector space $\mathbb{R}^3$ over $\mathbb{R}$.

iv) Give examples of two non-zero linear operators T_1, T_2 defined on a finite dimensional vector space V such that $T_1 \circ T_2 = 0$. 2+3+3+2

b) i) Find all maximal ideal of the ring $(\mathbb{Z}_{12},+,.)$.

ii) Let R be a ring with 1. Show that char R : n if and only if n is the least positive integer such that n1 = 0.

iii) Let U be the subspace spanned by the set of vectors {(2,0,1), (3,1,0)} and W be the subspace spanned by the set of vectors {(1,0,0), (0,1,0)} of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Then find a basis for the space $U \cap W$.

iv) If A and B are the matrices corresponding to two ordered bases of a finite dimensional vector space V, then show that A and B are similar matrices. 3+2+2+3

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